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Highest weights, projective geometry, and the classical limit: I. Geometrical aspects and the classical limit

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Abstract

This paper starts with a new proof that highest weight vectors for semi-simple Lie group representations can be characterised by quadratic equations, and finds the automorphism group of this quadratic variety. The idea is illustrated by various geometrical examples. Various generalisations to Clifford algebras and quantum groups are explored, as well as the relationship between geometry, second quantisation, and the classical limit. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [22] a characterisation of highest weight vectors was presented together with examples suggesting that it might be of wider significance for other classes of groups and even for some modular representations.

Theorem 1.1. *Let G be a compact semi-simple Lie group and V an irreducible G-module. A vector v in V is a highest weight vector for some maximal torus in G if and only if the cyclic G-submodule of $V \otimes V$ generated by $v \otimes v$ is irreducible.*

This paper and its sequel explore three main themes: geometrical examples of this theorem, some new examples of similar behaviour for other groups, and generalisations, particularly to Hopf algebras such as superalgebras and quantum groups. The motivation to return

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to the topic came partly from the general interest in Hopf algebras generated by quantum group theory, and partly from the realisation that such topics do interest geometers. The textbook of Fulton and Harris [18] has provided a beautiful introduction to the applications of representation theory in geometry, and the work of Zak on Severi varieties [35], illustrates how non-trivial these applications can be. A more recent appearance is in the study of harmonic maps [8]. It is the fact that the orbits of highest weight vectors are algebraic varieties which lends them a particular importance, and Theorem 1.1 displays the geometry of highest weight vectors more clearly than the normal definitions. It characterises the orbit of a highest weight vector as a homogeneous quadratic variety, for introducing the projection P onto the irreducible submodule of $V \otimes V$ whose highest weight is double that of V, the highest weight vectors are those satisfying the quadratic equation $P(v \otimes v) = v \otimes v$. (Chevalley observed this property of highest weight vectors for the classical compact semi-simple Lie groups, and the above general result was also proved independently by Lichtenstein [36].)

A number of classical geometrical configurations appear as special cases of highest weight quadrics, perhaps the most obvious arising from the irreducible representation of $SU(4)$ on $\wedge^2\mathbb{C}^4$. The highest weight vectors are those on the orbit of $e_1 \wedge e_2$, where the e_j form the natural basis of $\mathbb{C}^4$, and these are just the decomposable elements of $\wedge^2\mathbb{C}^4$. It is known that these may be characterised as the points of the Klein quadric, $\epsilon(v, v) = 0$ defined by the natural symmetric bilinear pairing

$$\epsilon : \wedge^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^4 \to \wedge^4\mathbb{C}^4 \cong \mathbb{C}.$$

Definition 1.1. For any group G and G-module V let us call a vector $v \in V$ *dominant* if $v \otimes v$ generates an irreducible submodule of the tensor product representation or, more generally, a submodule of minimal composition length.

Theorem 1.1 tells us that this is consistent with the use of dominant in the context of semi-simple Lie groups. We note, however, that any vector on the orbit of a highest weight vector is dominant, including the lowest weight vector. Since we are interested in homogeneous equations we work in the context of projective geometry, that is, for each G module V we consider $\mathbb{P}(V)$, and we shall describe rays as well as vectors as being dominant. In that context it makes no difference whether we work with ordinary or projective representations of G.

It was already noted in [22] that some group representations seem to require the consideration of tensor powers higher than the second, and these geometric examples also suggest that one might profitably consider other polynomial equations of arbitrary degree.

The author's original interest in these ideas was not geometrical, but came rather from the philosophy acquired at lectures of Mackey that the representation theory of Lie groups should not be very different from that of other locally compact groups. The representation theory of type I nilpotent separable locally compact groups offers some encouragement for this view, for, as shown independently by Howe [26] and Hannabuss [21], the irreducible representations of such groups are always monomial, just as in Kirillov's Lie group theory.

Geometric quantisation suggests that for Lie groups such representations, constructed using real polarisations, are at the opposite extreme to the holomorphic representations which tend to be associated with highest weight vectors. To consider the latter we first recall that each cyclic vector v for a square-integrable unitary representation U of a locally compact group G enables us to embed U into a subrepresentation of the left regular representation by taking a vector u in the representation space to the square-integrable function

$$g \in G \mapsto (T_v u)(g) = \lambda \langle U(g)v, u \rangle.$$

When λ is the formal dimension the intertwining map T_v is unitary and $T_v(U(g)v)$ provides a reproducing kernel for the subspace [15]. We could also describe this more geometrically by introducing the line bundle over the orbit of v whose fibre at each vector is the one-dimensional subspace which it spans and mapping u to the section $U(g)v \mapsto (T_v u)(g)U(g)v$. When this general construction is applied to familiar examples such as the irreducible representations of a compact semi-simple Lie group it is only for dominant v that the bundles and sections are holomorphic. (Then the reproducing kernel picks out the holomorphic subspace and provides a substitute for the Cauchy–Riemann equations.)

This suggests that the most useful realisations of representations of more general locally compact groups will require a careful choice of the vector v, and the dominant vectors or their generalisations to higher degree provide natural candidates to try. It is, of course, not easy to find which vectors are dominant for general groups G, since the definition of dominance presupposes a good knowledge of the representation theory of the group in question. Nonetheless, there are exceptions, and even when the representations are already known, the dominant vectors may lead to useful realisations of them. The notion of dominance can also be extended to representations of Hopf algebras for which tensor products of representations also make sense.

The paper is organised as follows. A new proof of Theorem 1.1 is given in Section 2, and is illustrated by various examples showing how familiar geometrical configurations arise naturally in this context. The variety of dominant vectors is generally quite rigid and in Section 3 it is shown that, with a few exceptions, the only linear transformations of V which preserve dominance are those arising from the action of the complexification of G. (This follows from a characterisation of the biholomorphisms on a flag manifold, which is proved in Appendix A.) Section 4 contains a demonstration that the dominant vectors for direct products are tensor products of dominant vectors.

The generalisation to higher degrees is introduced in Section 5. The finite symmetry groups of the regular solids discussed in Section 6 cast other familiar geometric configurations, the Kleinian singularities, into a new form on a conic in the projective plane rather than on the projective line. In Section 7 the relationship between the dominant vectors for symmetric groups and Young symmetrisers is considered, albeit rather inconclusively. Section 8 is devoted to convexity properties of dominant vectors for both their Lie algebras and their L^1 group algebras. This culminates in Theorem 8.2 which states that, with a few low-dimensional exceptions, a moment map derived from the conjugation action of the group on states projects pure states for an irreducible module of the group algebra onto the closed convex hull of the coadjoint orbit associated with the module. In the exceptional cases the image is just the orbit itself.

The geometrical properties of dominant vectors suggest a link between the classical limit of a quantum system and its second quantisation, and they play a crucial role in the notion of the classical limit of a quantum system and in particular, in a new proof of the Lieb–Simon theorem [47], which is presented in Section 9. The last two sections explore in the cases of Clifford algebras (Section 10) and quantum groups (Section 11) the generalisation of dominant vectors to Hopf algebras (for which tensor product representations can also be defined).

The sequel to this paper will concentrate on the case of infinite-dimensional groups and representations.

2. Compact semi-simple Lie groups

The proof of Theorem 1.1 given in [22] referred back to [14]; in this section we shall present a slightly more self-contained proof, that will provide us with some additional information. (The proof in [36] is totally different.)

Proof of Theorem 1.1. Let G be a compact semi-simple group and V a finite-dimensional irreducible G-module. Let $\{X_\alpha\}$ be a basis for the Lie algebra $\mathfrak{g}$ of G, let K denote the Killing form with components $K_{\alpha\beta} = K(X_\alpha, X_\beta)$, and let $K^{\alpha\beta}$ be the components of the dual inner product on $\mathfrak{g}^*$.

Using the summation convention, the dual Killing form

$$\begin{aligned} K^{\alpha\beta} X_\alpha \otimes X_\beta = \tfrac{1}{2}(K^{\alpha\beta}(X_\alpha \otimes 1 + 1 \otimes X_\alpha)(X_\beta \otimes 1 + 1 \otimes X_\beta) \\ - K^{\alpha\beta} X_\alpha X_\beta \otimes 1 - K^{\alpha\beta} 1 \otimes X_\alpha X_\beta) \end{aligned}$$

in the tensor product of the enveloping algebra is central and even a difference of Casimir elements. It therefore takes scalar values in irreducible G-modules. We may therefore deduce that, if $v \in V$ is such that $v \otimes v$ generates an irreducible submodule of $\otimes_S^2 V$ then

$$K^{\alpha\beta} X_\alpha v \otimes X_\beta v = \kappa v \otimes v,$$

for some scalar κ. As the action of the Lie algebra on the tensor product commutes with the dual Killing form, by applying enough negative root vectors we can obtain the same identity, but with the lowest weight vector ϕ replacing v. If the irreducible representation is labelled by the highest weight λ for some Cartan subgroup then $\phi \otimes \phi$ generates the irreducible with highest weight 2λ, and so, writing ρ for half the sum of the positive roots, we have

$$\kappa = -\tfrac{1}{2}(\langle 2\lambda, 2\lambda + \rho\rangle - 2\langle \lambda, \lambda + \rho\rangle) = -\|\lambda\|^2.$$

Taking the inner product of the original identity with v we obtain

$$K^{\alpha\beta}\langle v, X_\alpha v\rangle X_\beta v = \kappa \|v\|^2 v.$$

For any $u \in V$ we write $H_u = K^{\alpha\beta}\langle u, X_\alpha u\rangle X_\beta$ and then this reduces to the eigenvalue equation

$$H_v v = \kappa \|v\|^2 v.$$

Another inner product with v yields

$$\langle v, H_v v\rangle = \kappa\|v\|^4 = -\|\lambda\|^2\|v\|^4,$$

but the definition of H_u shows that for any $Y \in \mathfrak{g}$, $K(H_u, Y) = \langle u, Yu\rangle$, so that, in particular,

$$\|H_v\|^2 = |K(H_v, H_v)| = \langle v, H_v v\rangle = \|\lambda\|^2\|v\|^4.$$

Now consider the Cartan subalgebra of $\mathfrak{g}$ containing H_u as its first basis element. As u varies over the space $|\langle u, Hu\rangle|/\|u\|^2$, it is bounded above by $|\tilde{\lambda}(H_u)|$ (where $\tilde{\lambda}$ is the appropriate conjugate of λ), and the Cauchy–Schwarz–Bunyakovski inequality bounds this by $\|\tilde{\lambda}\|\|(H_u)\| = \|\lambda\|\|H_u\|$. But

$$\frac{|\langle v, Hv\rangle|}{\|v\|^2} = \|\lambda\|^2\|v\|^2 = \|\lambda\|\|H_v\|,$$

so that the bound is attained for $u = v$, which can only happen if both the inequalities are saturated and v is a highest (or lowest) weight vector for the chosen Cartan subalgebra. (We also see that $K(H_v, \cdot)$ must define the linear functional λ.) □

The above construction generalises to provide other useful information for very little extra effort. For a start we can read off the following useful inequality, which is equivalent to the Delbourgo–Fox minimal uncertainty theorem [14]. (This mathematical formulation, probably already known to those working in the area, found a place in the general theory of convexity established by Kostant [32], Atiyah [1], Guillemin and Sternberg [19,20] and Kirwan [30]; see also [49].) We shall discuss this further in Section 8.

Corollary 2.1. *In the above notation we have* $|K(H_u, H_u)| \le \|\lambda\|^2\|u\|^4$.

Proof. We know that

$$\|H_u\|^2 = |K(H_u, H_u)| = |\langle u, H_u u\rangle| \le \|\lambda\|\|H_u\|\|u\|^2.$$

The first and last elements of this inequality give $\|H_u\| \le \|\lambda\|\|u\|^2$, which when substituted gives the stated inequality. □

In another useful extension of the method, we may apply the element $\xi \in V^*$ to both sides of the original identity to obtain

$$K^{\alpha\beta}\xi(X_\alpha v)X_\beta v = \kappa\xi(v)v.$$

Writing $K(\xi) = K^{\alpha\beta}\xi(X_\alpha v)X_\beta$, this reduces to

$$K(\xi)v = \kappa\xi(v)v.$$

Thus v is an eigenvector of each of the elements $K(\xi)$. We also have

$$K(K(\xi), Y) = K^{\alpha\beta}\xi(X_\alpha v)K(X_\beta, Y) = \xi(X_\alpha v)Y^\alpha = \xi(Yv),$$

so we see that $Y \in \mathfrak{g}$ is orthogonal to $K(\xi)$ if and only if $\xi(Yv) = 0$. The orthogonal complement of

$$K(V^*) = \{K(\xi) \in \mathfrak{g}: \xi \in V^*\}$$

is therefore precisely

$$K(V^*)^\perp = \{Y \in \mathfrak{g}: Yv = 0\}.$$

This shows that v is an eigenvector for the whole of $\mathfrak{p} = K(V^*) + K(V^*)^-$. Moreover, $\mathfrak{p}$ is the whole subalgebra of elements having v as an eigenvector, for any two such elements must have a linear combination annihilating v. Since we have established that v is a highest weight vector for some Cartan subalgebra $\mathfrak{p}$ must be the corresponding parabolic subalgebra.

One advantage of this algebraic approach, besides its explicit derivation of the parabolic subalgebra $\mathfrak{p}$, is its potential to extend to some other cases such as affine Lie algebras or those Lie superalgebras which have non-degenerate Killing forms. This requires, however, knowledge of the structure theory of such algebras. In the case of the affine algebras, which are the Lie algebras of loop groups Map(S^1, G), the highest weight vectors are eigenvectors for a particular sort of borel subalgebra containing the constant functions in a borel subalgebra of G together with functions having only negative Fourier series. This is not the most general possibility (as we shall see later), but is the only one consistent with other useful properties.

For example, the dominant vectors in the natural representation of $SO(n)$ on $\mathbb{C}^n$ (for $n > 1$) are always determined by the single quadratic equation $v \cdot v = 0$ which gives the standard quadric in $\mathbb{CP}^{n-1}$. This may easily be seen by observing that the symmetric tensors split into the direct sum of the one-dimensional space spanned by the $SO(n)$-invariant inner bilinear form and its orthogonal complement, and choosing vectors v such that $v \otimes v$ lies in the latter subspace. (Being only of rank 1 $v \otimes v$ can never be in the one-dimensional space.) When $n = 3$ the dominant vectors for the Cartan subgroup of rotations about the third axis are the two circular points at infinity $(1, \pm i, 0)$.

In the case of the adjoint representation it is possible to give a neater form to the defining quadratics.

Corollary 2.2. *Let G be a compact semi-simple Lie group. The dominant vectors in the adjoint representation are those vectors $v \in \mathfrak{g}$ which satisfy*

$$[v, [v, z]] = \|\lambda\|^2 K(v, z)v,$$

for all $z \in \mathfrak{g}$, where λ is the highest root for some Cartan subgroup. Dominant vectors are always nilpotent elements of $\mathfrak{g}$, and satisfy $K(v, v) = 0$.

Proof. The equations for the dominant vectors derived above give in this case

$$K^{\alpha\beta}[X_\alpha, v] \otimes [X_\beta, v] = \kappa v \otimes v,$$

with $\kappa = -\|\lambda\|^2$. Taking the Killing product of the right-hand side with $y \otimes z \in \mathfrak{g} \otimes \mathfrak{g}$ gives just $K(y, v)K(z, v)$, and using the adjoint invariance of K the product with the left-hand side is

$$K^{\alpha\beta}K([X_\alpha, v], y)K([X_\beta, v], z)$$
$$= K^{\alpha\beta}K(X_\alpha, [v, y])K(X_\beta, [v, z]) = K([v, y], [v, z]) = -K(y, [v, [v, z]]).$$

Since these must be equal for all y and z we deduce the stated equation: $[v, [v, z]] = \|\lambda\|^2 K(v, z)v$. Setting $z = v$ tells us that $K(v, v)$ vanishes. Applying $\mathrm{ad}(v)$ to the equation shows that $\mathrm{ad}(v)^3 = 0$, from which the nilpotence of v follows. □

For example, the Lie algebra $\mathfrak{so}(3)$ is just $\mathbb{R}^3$ equipped with the usual vector product as Lie multiplication the dot product as Killing form. The equation for a dominant vector is therefore (for all $z \in \mathbb{C}^3$), $v \times (v \times z) = (v \cdot z)v$ or, equivalently, $(v \cdot v) = 0$, as already seen above.

When $G = SU(n)$ and the Lie algebra can be identified with self-adjoint operators on $\mathbb{C}^n$ having trace zero, one actually has $v^2 = 0$. For arbitrary vectors u and w, taking $z = uw^* - \langle w, u\rangle$ gives the constraint

$$v^2uw^* - 2vuw^*v + uw^*v^2 = \|\lambda\|^2\langle w, vu\rangle v.$$

Now taking u to be in the kernel of the nilpotent v, all terms but the last on the left vanish leaving $w^*v^2 = 0$ for all w, and this forces $v^2 = 0$. The remaining equality now reduces to $-2vzv = \|\lambda\|^2 K(v, z)v$, and actually forces v to have rank 1, which shows that $v = ab^*$ for orthogonal vectors a and b in $\mathbb{C}^n$. (We can explicitly identify this with the flag $\langle a\rangle \subseteq \langle b\rangle^\perp$.)

Representations of non-compact semi-simple Lie groups need not in general contain any dominant vectors, but when they do the same argument may be used to characterise those vectors as dominant. This works for the discrete series, but not for the principal series, where one must turn to the generalised dominant vectors defined in Section 7 of [23].

Although the characterisation of dominant vectors afforded by Theorem 1.1 requires neither the choice of a Cartan subgroup nor an ordering of the roots, the Weyl group and the half sum of the positive roots can be recovered geometrically. Suppose for simplicity that the stabiliser T of the ray $\langle v\rangle$ through the dominant vector v in $\mathbb{P}(V)$ is a maximal torus. Then the Weyl group W is just the group of G-isomorphisms of the variety of rays through dominant vectors $X \cong G/T$ with itself, since the G isomorphisms of G/T are of the form $gT \mapsto gnT$ for some n in the normaliser of T.

As a subset of $\mathbb{P}(V)$, X inherits a complex structure. We can therefore form the complex tangent space $\Theta_{\langle v\rangle}X$ which can be identified with the space $\mathfrak{n}$ of positive root vectors in the Lie algebra $\mathfrak{g}$ via the correspondence $X \in \mathfrak{g} \leftrightarrow X \cdot v/\mathbb{C}v$. (The quotient by $\mathbb{C}v$ allows for the fact that we are dealing with the tangents to the projective space.) As the action of T on $\Theta_{\langle v\rangle}E$ is described by the positive roots, $\omega = \det^{1/2}$ is the exponential of half the sum of the positive weights, ρ (it is also the highest weight of the spin action of T on spinors for the tangent space).

Any representation μ of T can be conjugated by $\sigma \in W$ to $\mu^{\sigma}(t) = \mu(ntn^{-1})$. This applies in particular to the action on the real tangent space $T_{\langle v\rangle}X$, and the conjugated action will, in general, no longer preserve the complex structure; the original complex structure J given by multiplication by i on the complex tangent space will now act as multiplication by $-\mathrm{i}$ on some subspace. The length of σ is defined as the dimension $l(\sigma)$ of this subspace.

Together with the restriction L to X of the tautological line bundle $V \to \mathbb{P}(V)$, W and ρ provide all the ingredients for the standard tools of the representation theory of semi-simple compact groups, and make it possible to discuss the Borel–Weil–Bott Theorem, Bernstein–Gel'fand–Gel'fand (BGG) resolutions, generalised Penrose transforms between different spaces of dominant vectors for G and in some cases, Ward transforms [2–5].

For any group it would be possible to stratify the representation space with strata V_k consisting of vectors v such that $v \otimes v$ generates a submodule with a composition series of length k, so that

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$$

3. Symmetries of the dominant vectors

The variety of dominant vectors is quite rigid in V as we see from Theorem 3.1. Before stating it, we note that as the representation space V is complex the action of the group G can be extended to an action of its complexification $G_{\mathbb{C}}$, which will clearly respect the quadratic constraints defining dominant vectors. The stabiliser in $G_{\mathbb{C}}$ of a dominant vector is a parabolic subgroup P. We shall use the notation of Baston and Eastwood [2] for Dynkin diagrams of parabolic subalgebras, but orient the Dynkin diagrams as in [6] (for G_2 this is the opposite direction to [2]).

Theorem 3.1. *Let G be a compact semi-simple Lie group and V an irreducible G module. Unless $G = B_r$ $(r > 1)$ and the Dynkin diagram for P is crossed on the first node, or $G = C_r$ $(r > 1)$ or G_2 and the Dynkin diagram for P is crossed on the last node, or $G = F_4$ and the Dynkin diagram for P is crossed on the third node, the connected subgroup of $PGL(V)$ which maps dominant rays to dominant rays is the image of $G_{\mathbb{C}}$.*

Proof. We have already noted that $G_{\mathbb{C}}$ maps dominant vectors to dominant vectors, so that the subgroup of $PGL(V)$ which preserves the variety of dominant rays is at least $G_{\mathbb{C}}$, and we need only show that under our hypotheses its connected component is no larger. As usual the space V can be realised as sections of the holomorphic line bundle L of the previous section by taking a vector $\psi \in V$ to the section s_{ψ} whose value at a dominant ray $\langle v\rangle$ is

$$s_{\psi}(\langle v\rangle) = \frac{\langle v, \psi\rangle}{\|v\|^2} v.$$

If $A \in GL(V)$ takes dominant vectors to dominant vectors then

$$s_{A^*\psi}(\langle v\rangle) = \frac{\langle v, A^*\psi\rangle}{\|v\|^2} v = \frac{\langle Av, \psi\rangle}{\|v\|^2} v = \frac{\|Av\|^2}{\|v\|^2} A^{-1} s_{\psi}(\langle Av\rangle).$$

This certainly defines a linear transformation provided that it preserves the subspace of holomorphic sections which are identified with V. For this A must define a biholomorphic transformation of the space of dominant vectors. By Theorem A.1, with the stated exceptions, this is just $G_{\mathbb{C}}$. □

Theorem A.1 could also be used to study what happens in the exceptional cases. (Theorem A.2 shows what happens for maximal parabolics of B_r, C_r and G_2. The fact that the flag manifold of F_4 is not a flag manifold of E_6 means that this case is more complicated.) Clearly the finite group of automorphisms of G acts on V only when its action on the Dynkin diagram preserves the labelling by the weights of the representation.

As an example of this theorem consider the five-dimensional irreducible representation of the rotation group on 3×3 trace-zero symmetric matrices. The quadric defining dominant vector v is given by $v^2 = 0$, so the only linear transformations of the five-dimensional space which preserve this quadric are the complex orthogonal transformations. Similarly the only linear transformations of a Lie algebra $\mathfrak{g}$ which preserve the quadric given by $\mathrm{ad}(v)^2 = \|\lambda\|^2 K(v, \cdot)v$ are those given by the adjoint action of the complex group $G_{\mathbb{C}}$.

4. Direct product groups

The irreducible representations of direct product groups are the tensor products of irreducible representations of the factors, so we might expect that their dominant vectors will also be tensor products of dominant vectors, and this is indeed the case.

Theorem 4.1. *Let V and W be irreducible representation spaces for the groups H and K, respectively, and $v \in V$ and $w \in W$ dominant vectors. Then $v \otimes w$ is a dominant vector for the tensor product representation of $G = H \times K$ in $V \otimes W$, and every dominant vector has this form. The same applies for projective representations.*

Proof. Clearly $v \otimes w$ generates a tensor product of irreducibles, which is itself irreducible. Conversely, any vector Ω in $V \otimes W$ can be decomposed as

$$\Omega = \sum_{\alpha} v_\alpha \otimes w_\alpha,$$

with $v_\alpha \in V$ and $w_\alpha \in W$. Identifying $(V \otimes W) \otimes_S (W \otimes W)$ with a subspace of $(V \otimes V) \otimes (W \otimes W)$ we have

$$\begin{aligned}\Omega \otimes \Omega &= \sum_{\alpha\beta} (v_\alpha \otimes w_\alpha) \otimes_S (v_\beta \otimes w_\beta) \\ &= \sum (v_\alpha \otimes_S v_\beta) \otimes (w_\alpha \otimes_S w_\beta) + \sum (v_\alpha \wedge v_\beta) \otimes (w_\alpha \wedge w_\beta).\end{aligned}$$

For irreducibility the second term must vanish, and this is only possible if either all the v_α are multiples of each other or all the w_α are multiples of each other. In either case the original expression for Ω can be reduced to a single term, so that we might as well write $\Omega = v \otimes w$. When Ω has this form we have

$$\Omega \otimes \Omega = (v \otimes v) \otimes (w \otimes w),$$

which generates an irreducible if and only if v and w generate irreducibles, that is they are dominant. □

One trivial geometric consequence of this arises when we use the isomorphism $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ to identify the natural representation of $SO(4)$ with the tensor product of natural representations of $SU(2)$. The quadric of dominant vectors in $\mathbb{CP}^3$ is identified with the product $\mathbb{CP}^1 \times \mathbb{CP}^1$, exhibiting its two families of generating lines.

5. Varieties of higher degree

We shall therefore consider representations of the central extension $\tilde{G}$ of G by $\mathbb{T}$, in which the centre acts on V as multiplication by itself considered as a subset of $\mathbb{C}$. The fact that $v \otimes v$ generates an irreducible subrepresentation of $V \otimes V$ can be interpreted as saying that the ray through v lies in a minimal G-invariant quadric in V. (In fact, minimality imposes a more flexible condition, since sometimes $v \otimes v$ generates not an irreducible but a representation with shortest composition series [22], or of least dimension, see the case of A_4 below.) We can therefore look more generally for minimal G-invariant projective subvarieties of $\mathbb{P}(V)$. (We could generalise the ideas further and study G-invariant subvarieties of any Grassmannian $Gr_k(V)$ or even flag manifolds.) Equivalently we may look for the corresponding maximal G-invariant radical ideals in $\mathbb{C}[V]$ defining these varieties. The group action on polynomials is the transpose of its action on V; that is, $g \cdot f(v) = f(g^{-1}v)$. (The centre of the central extension helps distinguish polynomials of different degree. One could work over fields other than $\mathbb{C}$, though, of course, there may be some difficulties if the field is not algebraically closed, and the centre will certainly not distinguish degrees in general. moreover, for fields with finite characteristic one needs to use not the whole polynomial ring, but its quotient by a Frobenius twisted ideal [16,33].) In finite dimensions such maximal ideals always exist.

Theorem 5.1. *Let V be a finite-dimensional module for a group G. Then there exist maximal G-invariant radical ideals properly contained in $\mathbb{C}[V]$.*

Proof. Consider an increasing chain of G-invariant radical ideals properly contained in $\mathbb{C}[V]$. Since $\mathbb{C}[V]$ is Noetherian it becomes stationary at a certain maximal element. Thus by a Zorn's lemma argument maximal G-invariant radical ideals exist. (These may be $\{0\}$, as for the natural representation of $SU(n)$.) □

We remark that the deformed polynomial rings (that is q-symmetric tensor algebras) obtained for quantum groups have ordinary polynomial rings for their associated graded products, so these are also Noetherian, a fact which is useful in the context of quantum groups.

As long as one is interested only in compact semi-simple Lie groups it is unnecessary to consider degrees higher than 2, since any tensor power of a highest weight vector also has highest weight, and so higher powers add no new information. Put another way the quadrics generate the whole ideal which vanishes on the orbit of a highest weight vector. For deeper consideration of this we refer to Kostant's theorem in [34] (see also [24]) and to the generalisation of Kostant's theorem to Schubert varieties by Ramanathan [45]. Linear polynomials simply pick out irreducible subspaces if the representation itself is not irreducible, and thus perform harmonic analysis.

The fact that tensor powers of highest weight vectors for semi-simple Lie groups are also highest weight vectors is reflected in some well-known geometrical facts. The map $v \in V \mapsto v^{(k)} \in V^{(k)} = \bigotimes_S^k V$ induces a degree k map of the projective spaces which must map the variety of dominant vectors in $\mathbb{P}(V)$ to the variety of dominant vectors in $\mathbb{P}(V^{(k)})$. Every non-zero vector in the natural representation of $SU(n+1)$ is dominant, so there is a degree k map from $\mathbb{CP}^n$ to the quadratic variety of dominant vectors in $\mathbb{CP}^D$ where $D = \binom{n+k}{k} - 1$. In particular when $n = 1$ and $k = 2$ we obtain the quadratic parametrisation of a conic in the $\mathbb{CP}^2$ by $\mathbb{CP}^1$, and when $n = 1$ and $k = 3$ we obtain the twisted cubic identification of the intersection of a net of quadrics in $\mathbb{CP}^3$.

6. Symmetries of regular solids

The analysis of Section 2 can also be extended to a more general study of the three-dimensional representations of the symmetries of the Platonic solids. Under rotations the symmetric tensor product of $V \cong \mathbb{C}^3$ with itself decomposes into the one-dimensional subspace spanned by the Euclidean bilinear form and its five-dimensional complement. Under finite subgroups the symmetric tensor product often decomposes further. For instance under the cube–octahedral symmetry group S_4 the five-dimensional irreducible splits further into a two-dimensional and a three-dimensional piece. With respect to an orthonormal basis e_1, e_2, e_3 for V, aligned along the edges of the cube the three-dimensional irreducible piece is spanned by $e_j \otimes_S e_k$ with $j \neq k$, the two-dimensional piece by $e_j \otimes e_j - e_3 \otimes e_3$, $j = 1$, 2, and the one-dimensional piece by $e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$.

It is easy to check that there are no longer any vectors v in V whose tensor square generates an irreducible, but $v \otimes v$ can have components in just two of the irreducible components, avoiding the third. There are thus three possible ways in which v can be dominant, by being orthogonal to the three-dimensional, the two-dimensional or the one-dimensional irreducible. The condition for being orthogonal to the three-dimensional piece can be expressed in terms of the coordinates (v_1, v_2, v_3) of v with respect to the basis as

$$v_1 v_2 = 0, \quad v_2 v_3 = 0, \quad v_3 v_1 = 0.$$

These equations determine the three coordinate axes or equivalently the axes through the face centres of the cube or the vertices of the octahedron. To be orthogonal to the two-dimensional component v must satisfy

$$v_1^2 = v_2^2 = v_3^2,$$

an equation which picks out the four long diagonals of the cube or the axes through the face centres of the octahedron. (It is worth noting that these four points determine a quadrilateral in $\mathbb{CP}^2$ whose self-polar triangle is given by the three points determined by the previous equation. One thus recovers the "complete quadrilateral" of plane projective geometry as a configuration of dominant vectors for the cube–octahedral group.) Finally, orthogonality to the one-dimensional irreducible is equivalent to the identity

$$F_2 \equiv v \cdot v = v_1^2 + v_2^2 + v_3^2 = 0,$$

which picks out the dominant vectors v for the rotation group.

One may distinguish between rotationally dominant vectors satisfying $v \cdot v = 0$ by going to higher powers. Common sense suggests that, since we already have one constraint in the projective plane, we can impose at most one more independent constraint without overdetermining things. This intuition is confirmed by calculation, which shows that in the symmetric cube it is possible to constrain v so that $v^{(3)} = v \otimes v \otimes v$ is orthogonal to the unique one-dimensional subrepresentation, but not to any other irreducible component. The condition for this is given by the cubic equation

$$F_3 \equiv v_1 v_2 v_3 = 0.$$

Instead of imposing this constraint one could require $v^{(4)}$ to be orthogonal to the one-dimensional constituent of the symmetrised fourth tensor power, which leads to the quartic constraint

$$F_4 \equiv v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2 = 0,$$

or equivalently, in view of our quadratic constraint, $v_1^4 + v_2^4 + v_3^4 = 0$. Another alternative to imposing either of these cubic and quartic constraints would be the sextic constraint

$$F_6 \equiv (v_1^2 - v_2^2)(v_2^2 - v_3^2)(v_3^2 - v_1^2) = 0.$$

One readily calculates the syzygy

$$F_6^2 + 4F_4^3 + 27F_3^4 = 24F_2 F_3^2 F_4 - 4F_2^3 F_3^2 + F_2^2 F_4^2.$$

As we recalled in Section 5 the quadric $v \cdot v = 0$ in $\mathbb{CP}^2$ can be identified with $\mathbb{CP}^1$ by a quadratic parametrisation, for example,

$$v_1 = \frac{i}{2}(z_1^2 + z_2^2), \qquad v_2 = \frac{1}{2}(z_1^2 - z_2^2), \qquad v_3 = z_1 z_2,$$

where $z = z_1/z_2$ is the variable on $\mathbb{CP}^1$. In terms of z_1 and z_2 the cubic quartic and sextic constraints can be written as

$$z_1 z_2 (z_1^4 - z_2^4) = 0, \quad z_1^8 + 14 z_1^4 z_2^4 + z_2^8 = 0, \quad z_1^{12} - 33 z_1^8 z_2^4 - 33 z_1^4 z_2^8 + z_2^{12} = 0,$$

(cf. [31, Chapter I, Sections 10–12, Eqs. (40), (43), (52)]. On the Riemann sphere they define, respectively, the six vertices of an octahedron, the eight vertices of a cube and the 12 midpoints of the edges of the cube or octahedron. When z_1, z_2 are substituted, the syzygy simplifies because its right-hand side vanishes identically, leaving the usual identity [31, Eq. (53)].

There are no G-invariant subsets of these varieties, so one has amongst the maximal G-invariant ideals

$$\langle v_1v_2, v_2v_3, v_3v_1\rangle,\quad \langle v_1^2-v_2^2, v_2^2-v_3^2, v_3^2-v_1^2\rangle,\quad \langle F_2, F_3\rangle,\quad \langle F_2, F_4\rangle,\quad \langle F_2, F_6\rangle.$$

One may similarly find equations for the vertices of the icosahedron using the icosahedral group A_5. Another interesting class of examples is provided by the $(n-1)$-dimensional representation of S_n or A_n, associated with the partition $[n-1, 1]$, which has a convenient realisation as the restriction of the natural permutation representation of S_n on $V \cong \mathbb{C}^n$ to the invariant subspace of vectors such that $v_1 + v_2 + \cdots + v_n = 0$. The permutation action on $V \otimes_S V$ preserves its diagonal and off-diagonal parts. The diagonal part breaks up into a trivial submodule where all diagonal entries Z_{ii} of $Z \in V \otimes_S V$ are equal, and its orthogonal complement where $\sum_i Z_{ii} = 0$. The components of the off-diagonal part are most easily specified by describing their complements, and to do this we introduce $\sigma_2 = \sum_{i<j} Z_{ij}$, and $\sigma_2(k)$ which is the sum of Z_{ij} over $i < j$ omitting the index k. There is a trivial part whose complement is given by $\sigma_2 = 0$, an $(n-1)$-dimensional part whose complement is given by $\sigma_2(j) = \sigma_2(k)$ for all j and k, and (provided that $n > 3$) a $\frac{1}{2}n(n-3)$-dimensional piece, corresponding to the partition $[n-2, 2]$, whose complement is defined by the equations $Z_{ij} + Z_{kl} = Z_{il} + Z_{jk}$ for all distinct i, j, k and l. One can easily check that the condition for orthogonality to the $(n-1)$-dimensional representation can be written as $\sigma_2(j) = (n-2)\sigma_2/n$ for all j, and for orthogonality to the $\frac{1}{2}n(n-3)$-dimensional representation as

$$Z_{jk} = \frac{2}{n-1}\sigma_2 - \frac{1}{n-2}[\sigma_2(j) + \sigma_2(k)]$$

for all $j \neq k$. When $Z = v \otimes v$ the condition $\sum v_i = 0$ ensures that

$$\sum v_i^2 = -2\sigma_2, \qquad \sigma_2(k) = \sigma_2 + v_k^2,$$

so that the diagonal and off-diagonal parts are coupled, and consequently only one copy of the one-dimensional and $(n-1)$-dimensional representations occur in the symmetric square and the composition series for the subspace generated by $v \otimes v$ can be found by considering the off-diagonal part alone.

The equations $\sigma_2(j) = \sigma_2(k)$ are now equivalent to $v_j^2 = v_k^2$, so that $v_j = \pm v_1$ for all j. For odd n this gives $\sum v_k$ as a non-zero multiple of v_1, and this is inconsistent with the vanishing of that sum except in the trivial inadmissible case when $v = 0$. This means that $v \otimes v$ can never be orthogonal to the $(n-1)$-dimensional irreducible when n is odd. For even n such orthogonality is possible provided that there are equal numbers of positive and negative components.

Orthogonality to the $\frac{1}{2}n(n-3)$-dimensional piece gives the equations $v_iv_j + v_kv_l = v_iv_l + v_jv_k$ which can be rearranged as $(v_i - v_k)(v_j - v_l) = 0$. Since the sum of the components vanishes and $v \neq 0$, the components cannot all be the same. If $v_i \neq v_k$ then the above identity shows that all components other than v_i and v_k must be the same. Moreover, the vanishing of $(v_i - v_j)(v_k - v_l)$ shows that one of v_i and v_k takes the same value, in other words all but one component is the same and v must be a multiple of $(n-1, -1, -1, -1, -1)$ or some permutation of it. Explicit calculation then shows that then $v \otimes v$ is not orthogonal to either of the other irreducible pieces, so that $v \otimes v$ can at most be orthogonal to one irreducible. The condition for orthogonality to the trivial summand is satisfied by $v = (1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$ (and multiples and permutations) where $\lambda^n = 1$ but $\lambda \neq 1$. These vectors actually satisfy the hierarchy of equations

$$v_1^r + v_2^r + \cdots + v_n^r = 0$$

for $r = 1, \ldots, n-1$.

When $n = 4$ we can recover the dominant vectors in the three-dimensional representation of the cube–octahedral group, S_4 discussed above. When $n = 5$ we can deduce that there are two classes of quadratic dominant vectors for the icosahedral group A_5: those which are multiples and permutations of $(4, -1, -1, -1, -1)$, and those for which σ_2 vanishes. The latter class includes the multiples and permutations of $(1, \lambda, \lambda^2, \ldots, \lambda^4)$, where $\lambda^5 = 1$ but $\lambda \neq 1$, which not only lie in Klein's canonical surface $\sigma_2 = 0$, or, equivalently,

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 = 0,$$

but also satisfy the cubic equation

$$v_1^3 + v_2^3 + v_3^3 + v_4^3 + v_5^3 = 0,$$

which defines Clebsch's diagonal surface, and the quartic Bring equation [31, Section II.1]

$$v_1^4 + v_2^4 + v_3^4 + v_4^4 + v_5^4 = 0.$$

7. Symmetric groups

It would be interesting to find the dominant vectors of other representations of the symmetric groups. More precisely, given a Young tableau for S_n we let G_R be the subgroup stabilising its rows and G_C the subgroup stabilising its columns. We let S be the characteristic function of G_R, that is the function which takes the constant value 1 on G_R and vanishes outside it, and A be the function which coincides with the alternating character on G_C and vanishes outside. These may be considered as elements of the convolution group algebra of S_d and their product AS is the Young symmetriser. Considered as a vector in the regular representation AS lies in and generates an irreducible component associated with the Young tableau (see [28]). Actually a right-hand factor of S in an element of the group algebra tells us that the element is fixed under the right action of G_R and so can be identified with an element of the submodule of the regular representation induced from the trivial representation of G_R. One might hope that AS would be a dominant vector for that irreducible, but we shall show that in general it is not.

For example, labelling representations by the associated partitions, consider the module $[n-k, 1^k] = \wedge^k[n-1, 1]$, for $1 \leq k \leq n$. This can be realised as the submodule of the exterior power of the permutation module $\wedge^k\mathbb{C}^n = \wedge^k[n-1, 1] \oplus \wedge^{k-1}[n-1, 1]$ which is annihilated by inner multiplication by $e = (1, 1, \dots, 1)/n$ (see [18, Section 3.2]). The row stabiliser $G_R = S_{n-k}$ permutes $\{1, \dots, n-k\}$, whilst the column stabiliser $G_C = S_{k+1}$ permutes $\{1, n-k+1, \dots, n\}$. With respect to the natural basis $\{e_1, \dots, e_n\}$ for the permutation module, the vector $e_{n-k+1} \wedge e_{n-k+2} \wedge \cdots \wedge e_n$ is an eigenvector for a subgroup $S_k \times S_{n-k}$ (acting on the last k and first $n-k$ indices, respectively), and using the imprimitivity theorem $\wedge^k[n-1, 1]$ can be identified with the representation induced from the product of the alternating representation of $S_k \subseteq G_C$ and the trivial representation of $G_R = S_{n-k}$. The element S is fixed by the action of G_R and so can be naturally identified with the vector $e_n = (0, \dots, 0, 1)$. The alternating element A factorises into a product A_1A_2 of the alternating element A_2 for the smaller subgroup S_k which centralises G_R, and an alternating element on coset representatives for S_{k+1}/S_k. The factor A_2S tells us that the vector $AS = A_1A_2S$ lies in the induced representation space just identified with $\wedge^k\mathbb{C}^n$, and A_2S itself can be identified with $e_{n-k+1} \wedge e_{n-k+2} \wedge \cdots \wedge e_n$. It is now easy to see that $AS = A_1A_2S$ is identified with $(e_{n-k+1} - e_1) \wedge (e_{n-k+2} - e_1) \wedge \cdots \wedge (e_n - e_1)$. Similarly SAS is identified with $(e_{n-k+1} - e) \wedge (e_{n-k+2} - e) \wedge \cdots \wedge (e_n - e)$. The vector $SAS \otimes SAS$ generates the S_n-module induced from the trivial representation of $S_k \times S_{n-k}$, which is the direct sum of the irreducible representations $[n-j, j]$ from $j = 0$ to $[\frac{1}{2}n]$.

When we specialise to the case of $k = 1$, SAS is identified with a multiple of $(e_n - e)$, which we saw was a dominant vector for $[n-1, 1]$. On the other hand, AS is identified with $e_n - e_1 = (-1, 0, \dots, 0, 1)$, which is in the irreducible submodule $[n-1, 1]$, but not dominant. (In the case of S_3, SAS is not dominant either.)

8. Convexity

Theorem 1.1 and Corollary 2.1 have the dual version:

Theorem 8.1. *The set of functionals $f_u \in \mathfrak{g}^*$ defined by $f_u(Y) = \langle u, Yu\rangle$ for unit vectors u in the space of the irreducible representation of a compact semi-simple Lie group with highest weight λ form the ball*

$$K^*(f_u, f_u) \leq \|\lambda\|^2,$$

where K^* is the dual Killing form on $\mathfrak{g}^*$, and equality occurs if and only if v is a dominant vector.

Proof. In the notation of the proof of Theorem 1.1, $f_u = K(H_u, \cdot)$, so that this is just a restatement of Corollary 2.1 and Theorem 1.1. □

In this section we shall show how this is linked to another well-known use of convexity in the study of group representations. We recall [15, Section 13.6] that the matrix element $\omega(g) = \langle v, g\cdot v\rangle$ defines a positive-definite function on the group G (that is, for any choice of $x_1, \ldots, x_n \in G$, $\omega(x_j^{-1}x_k)$ is a positive matrix). It is normalised in the sense that $\omega(1) = 1$. It is known that the normalised positive-definite functions form a closed convex set $\mathcal{P}(G)$ in which ω is extreme if and only if v generates an irreducible submodule of V. The matrix element corresponding to $v\otimes v$ is

$$\omega^{(2)}(g) = \langle v\otimes v, g\cdot v\otimes g\cdot v\rangle = \omega(g)^2,$$

and so v is dominant if and only if this is extreme in $\mathcal{P}(G)^2$. (We shall take this as the definition of dominance for unitary representations, as the extremality of $\omega^{(2)}$ is the cleanest analytic formulation of the requirement that $v\otimes v$ generate a submodule with minimal composition series even when that is not irreducible.)

Definition 8.1. A positive-definite function $\omega \in \mathcal{P}(G)$ is said to be dominant if its square is extreme in $\mathcal{P}(G)^2$.

We could incorporate higher tensor powers by using higher powers of ω, or even incorporate all at once by considering $\exp(\omega)$ (unfortunately, this latter idea is slightly arbitrary, since we could equally well use $\exp(t\omega)$ for any positive real t).

Although this gives a neat formulation of dominance, it is less useful in practice. Nonetheless there are some cases in which it does give a direct answer. For example, consider the Klein four-group generated by elements a and b of order 2, with $c = ab = ba$ the remaining non-identity element. This has a two-dimensional projective representation given by

$$a\mapsto\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\qquad b\mapsto\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad c\mapsto\begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}.$$

(The multiplier σ has square 1 and causes no serious modification of our discussion above, except that ω must now be σ-positive, that is $\omega(x_j^{-1}x_k)/\sigma(x_j, x_j^{-1}x_k)$ is a positive matrix.) The normalised σ-positive-definite functions ϕ are determined by their values at a, b and c and these satisfy

$$-\phi(a)^2 - \phi(b)^2 - \phi(c)^2 \le 1,$$

with equality for the extreme points. Their squares $\phi^{(2)}$ lie in the tetrahedron

$$-\phi^{(2)}(a) - \phi^{(2)}(b) - \phi^{(2)}(c) \le 1,\quad -\phi^{(2)}(a),\ -\phi^{(2)}(b),\ -\phi^{(2)}(c) \ge 0,$$

whose non-trivial extreme points are given by functions $\phi^{(2)}$ which take the value -1 at one of a, b, c and vanish on the other two. The corresponding vectors v are those which are eigenvectors of one of a, b or c.

The extreme functions $\mathcal{P}_e(G)$ of a compact group G break up into connected components, $\mathcal{P}_e^V(G)$ labelled by equivalence classes of irreducible representations, V. Each component is finite-dimensional, since it contains just the functions $\langle v, g\cdot v\rangle$ with v a unit vector in that particular irreducible G-module V. Clearly there is a one–one correspondence between these functions and the points $\langle v \rangle$ in $\mathbb{P}V$. When G is a compact semi-simple Lie group these observations mean that each connected component has a symplectic structure pulled back from $\mathbb{P}V$. (It can be written in more algebraic terms by defining $a \in C(G)$ the convolution product function $x \mapsto (\omega * a)(x)$. The tangent space can then be identified with the $\omega * a$ which vanish at the identity, and the symplectic form is $s_\omega(\omega * a, \omega * b) = \mathrm{Im}(\omega(a^* b))$.) Because the representations are finite-dimensional the vectors are all analytic and we can define $p_\omega \in \mathfrak{g}^*$ as the derivative

$$p_\omega(X) = \mathrm{i}\, \mathrm{d}\omega[\exp(tX)]/\,\mathrm{d}t|_{t=0} = \mathrm{i}\langle v, X \cdot v \rangle,$$

(the factor of i being inserted to make it real-valued).

The group G acts on $\mathcal{P}_\mathrm{e}(G)$ by conjugating the argument, and it is easy to check that this action preserves the symplectic form and that $\omega \mapsto p_\omega$ is the moment map. Combining what we know about dominant vectors in the context of Lie groups and algebras, we obtain the following result:

Theorem 8.2. *Let G be a compact semi-simple Lie group. The extreme positive functions $\mathcal{P}_\mathrm{e}^V$ associated to an irreducible module V form a symplectic G-manifold. The moment map p_ω sends dominant vectors to the coadjoint orbit $\mathcal{O}^V$ associated to V. If none of the dominant weights of V differ by a root then the image $p_\omega(\mathcal{P}_\mathrm{e}^V)$ is the closed convex hull of the orbit $\mathcal{O}^V$.*

Proof. We know that dominant functions go to the linear functionals defined by highest weight vectors, and these are known to correspond to the coadjoint orbit. We also know that the image of all positive-definite functions is convex, but we wish to show this for the extreme functions. Now the continuation of the line segment joining any two pure states will meet the orbit in a pair of dominant states. So it suffices to show that any convex combination of elements of $\mathcal{O}^V$ are in the image of $\mathcal{P}_\mathrm{e}^V$ under the moment map. Equivalently given any dominant vectors $v_0, v_1, \ldots, v_n$, and any $t_0, t_1, \ldots, t_n \in (0, 1)$ which add up to 1, set $v = \sqrt{t_0} v_0 + \sqrt{t_1} v_1 + \cdots + \sqrt{t_n} v_n$. Now for any $X \in \mathfrak{g}$ and all j and k, the inner product $\langle v_j, X \cdot v_k \rangle$ so that we have

$$\sum \langle v, X \cdot v \rangle = \sum t_j \langle v_j, X \cdot v_j \rangle,$$

so that this convex combination can certainly be achieved by an extreme function. □

The convexity of the image mirrors the well-known fact that the weights for a chosen Cartan subgroup all lie in the convex hull of the Weyl orbit of the highest weight. However, the result is not true without the assumption on weights, because, for instance, all vectors in the natural representation of $U(n)$ are dominant, and so the image $p_\omega(\mathcal{P}_\mathrm{e}^V)$ will just consist of the image of the dominant states, that is the orbit $\mathcal{O}^V$ itself. A more detailed investigation of the low-dimensional cases in which some dominant weights do differ by roots has been given by Sjamaar [48].

Further insight into the relationship between Lie group and Lie algebra convexity can be obtained by the following observations. Since ω is normalised we can write $p_\omega(X) = \mathrm{i}\lim_{s\to 0}\ln[\omega(\exp(sX))]/s$ and, taking $s = 1/n$, we see that

$$\lim_{n\to\infty} \mathrm{i}n\ln[\omega(\exp(X/n))] = p_\omega(X).$$

(For a quantum mechanical system with Hamiltonian $-\mathrm{i}X$ the above argument is mathematically identical to the derivation of the quantum Zeno effect, in which frequent measurement inhibits evolution.) We have already remarked that tensor powers of highest weight vectors also generate irreducibles, so that $\omega^{(n)}(g) = \omega(g)^n$ is actually extreme for all $n \geq 1$, whence, by the concavity of logarithms, $p_\omega(X)$ is a limit of extreme functions.

There has been a lot of recent interest in the asymptotic behaviour of representations as the highest weight increases (see the work by Moreno [39–43] Cahen et al. [9–11,46] and Bowes and Hannabuss [7]). The basic motivation is the "correspondence principle" that classical mechanics is the asymptotic limit of quantum theory for large quantum numbers. Lichtenstein has proposed using the asymptotic behaviour of tensor powers of a vector v to pick out particular vectors [37].

9. The classical limit

The set of rays through dominant vectors form a projective variety X in $\mathbb{P}(V)$ which inherits a symplectic structure from the Kähler form on projective space. It therefore provides an analogue of the classical phase space of the system. (This is familiar in the case of the coherent states discussed in Sections 2 of [23].) The space of polynomials $\mathbb{C}[V]$ is isomorphic to the symmetric tensor algebra $\bigotimes_S V^*$, where V^* denotes the space dual to V. We shall denote the isomorphism by $p \mapsto \tilde{p}$. When V is a Hilbert space one has a natural antilinear identification of V and V^*, and the above considerations lead one to an interpretation of the polynomials in terms of Fock space. In the dual $(\bigotimes_S V^*)^*$ one has the coherent states

$$\exp(v) = \sum_{r=0}^{\infty} \frac{v^{\otimes r}}{r!},$$

(where $v^{(r)}$ denotes the r-fold tensor product of v with itself) which have the property that

$$\langle \exp(v), \tilde{p}\rangle = p(v).$$

In this way the classical phase space is recovered by consideration of its second quantisation. In some ways this is not too surprising since the second quantisation combines some features of both the particle (classical) and wave (quantum) pictures.

We shall now give a rather more concrete application to the classical limit by giving a new proof of the crucial step (Corollary A.2.7) in Simon's generalisation of Lieb's limit theorem for quantum and classical partition functions [38,47].

Theorem 9.1. *Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$, π an irreducible unitary representation of G and associated representation of $\mathfrak{g}$ on a space V, v a dominant vector in V, whose weight with respect to the stabilising Cartan subgroup is λ, and $P(g)$ the projection onto $\pi(g)v$. Then, writing $|G|$ for the Haar measure of G, for all $X \in \mathfrak{g}$*

$$(|G|\langle\lambda, \lambda\rangle)\pi(X) = \dim(V)\langle\lambda, \lambda + 2\rho\rangle \int_G \mathrm{tr}[P(g)\pi(X)]P(g)\,\mathrm{d}g.$$

Proof. We first note that for any A in $\mathcal{L}(V)$

$$A \mapsto \int_G \mathrm{tr}(P(g)A)P(g)\,\mathrm{d}g$$

commutes with the representation $\mathrm{ad}_\pi : A \mapsto \pi(x)A\pi(x)^*$ of G on $\mathcal{L}(V)$. This means that it does not mix inequivalent subrepresentations of ad_π, and in particular, operators of the form $A = \pi(X)$ get mapped to other operators transforming with the adjoint representation.

We next note that for X and Y in $\mathfrak{g}$ the element

$$\int_G \mathrm{ad}(g)^{-1}X \otimes \mathrm{ad}(g)^{-1}Y\,\mathrm{d}g$$

is G-invariant in $\mathfrak{g} \otimes \mathfrak{g}$. For a semi-simple group the only such elements are multiples of the dual Killing element. In terms of an orthonormal basis $\{X_\alpha\}$ and the notation of Appendix A we therefore have

$$\int_G \mathrm{ad}(g)^{-1}X \otimes \mathrm{ad}(g)^{-1}Y\,\mathrm{d}g = \gamma K^{\alpha\beta} X_\alpha \otimes X_\beta,$$

for some $\gamma \in \mathbb{C}$. To find γ we calculate the trace in the representation $\pi \otimes \pi$, to get

$$|G|\,\mathrm{tr}(\pi(X)\pi(Y)) = \gamma\,\mathrm{tr}(K^{\alpha\beta}\pi(X_\alpha)\pi(X_\beta)) = \gamma\dim(V)C_\pi,$$

where C_π denotes the eigenvalue of the Casimir operator $K^{\alpha\beta}\pi(X_\alpha)\pi(X_\beta)$. Taking the matrix element of the original identity with respect to $v \otimes v$ in the representation $\pi \otimes \pi$, we have

$$\int_G \mathrm{tr}(P(g)\pi(X))\,\mathrm{tr}(P(g)\pi(Y))\,\mathrm{d}g = \gamma\langle v \otimes v, K^{\alpha\beta}\pi(X_\alpha) \otimes \pi(X_\beta)v \otimes v\rangle.$$

So far everything is true for arbitrary v but when v is dominant it is an eigenvector for the polarised Casimir element (with eigenvalue $\tilde{C}_\pi$, say), so that the right-hand side reduces to $\gamma\tilde{C}_\pi$. Combining results we have

$$\int_G \mathrm{tr}(P(g)\pi(X))\mathrm{tr}(P(g)\pi(Y))\,\mathrm{d}g = \frac{|G|\tilde{C}_\pi\,\mathrm{tr}(\pi(X)\pi(Y))}{\dim(V)C_\pi}.$$

Since there is no mixing between the adjoint and other components of ad_π we conclude that

$$\int_G \mathrm{tr}(P(g)\pi(X))P(g)\,\mathrm{d}g = \frac{|G|\tilde{C}_\pi}{\dim(V)C_\pi}\pi(X),$$

from which the result follows on substituting the known values $C_\pi = -\langle\lambda, \lambda + 2\rho\rangle$ and $\tilde{C}_\pi = -\langle\lambda, \lambda\rangle$. □

The interesting feature of this proof is the fact that it is precisely the quadratic characterisation of the dominant vector which makes it work. An analogous formula for Heisenberg groups reduces to a well-known relationship between the projections and the representation.

10. Pure spinors and twistors

The idea of dominant vectors characterised by the properties of their tensor products is not limited to groups, for it can be propounded for any Hopf algebras. (For the rest of the paper we shall concentrate on the case of quadratic constraints.) Amongst the interesting examples of Hopf algebras which are not group algebras are the Lie superalgebras, where one works with $\mathbb{Z}_2$-graded tensor products of graded representations [29]. We have already noted that the proof of Theorem 1.1 can be adapted to some cases of semi-simple superalgebras with a non-degenerate Killing form. (For the rest of the paper we shall concentrate on the case of quadratic constraints, to which that argument applied.) A rather different example is provided by the Clifford algebras. Let V be a finite-dimensional real inner product space. Then the Clifford algebra $\mathrm{Cliff}(V)$ is the unital complex algebra generated by V subject to the relations $v^2 = \langle v, v\rangle$ for all $v \in V$. For simplicity let us deal with the case when V is even-dimensional. Then $\mathrm{Cliff}(V)$ has a unique irreducible representation which may be realised on the exterior algebra $\wedge W = \bigoplus \wedge^k W$, where W is a complex vector space whose direct sum with its conjugate $\overline{W}$ gives the complexification $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. Since any element of the complexification $V_\mathbb{C} \cong W \oplus \overline{W}$ can be written as $w_1 + w_2$ where $w_1 \in W$ and $w_2 \in \overline{W}$, the action of $\mathrm{Cliff}(V)$ is determined by that of W and $\overline{W}$. We let w_1 act by exterior multiplication by w_1 and w_2 act by interior multiplication with w_2.

The Hopf comultiplication sends v to $\delta(v) = v\widehat{\otimes}1 + 1\widehat{\otimes}v$, where $\widehat{\otimes}$ denotes the graded tensor product.

Theorem 10.1. *The dominant vectors in $\wedge W$ are the pure spinors.*

Proof. A general element of $\wedge W$ may be written as $u = u_0 + u_1 + \cdots + u_n$, where $u_k \in \wedge^k W$. The repeated action of $\overline{W}$ by inner multiplication on $\wedge W \widehat{\otimes} \wedge W$ must eventually reduce any element $u\widehat{\otimes}u$ to a multiple of $1\widehat{\otimes}1$. The space generated by $u\widehat{\otimes}u$ must therefore contain $1\widehat{\otimes}1$, and indeed be the same space as generated by that element.

On the other hand repeated exterior multiplication by elements of W gives

$$\delta(w_1)(1\widehat{\otimes}1) = w_1\widehat{\otimes}1 + 1\widehat{\otimes}w_1,$$

$$\delta(w_1)\delta(w_2)(1\widehat{\otimes}1) = (w_1 \wedge w_2)\widehat{\otimes}1 + w_1\widehat{\otimes}w_2 + w_2\widehat{\otimes}w_1 + 1\widehat{\otimes}w_1 \wedge w_2,$$

and so on. In general one gets

$$\delta(w_1)\delta(w_2)\cdots\delta(w_p)(1\widehat{\otimes}1) = \sum(w_I)\widehat{\otimes}(w_{I'}),$$

where $I \cup I'$ expresses $\{1, 2, \ldots, p\}$ as a disjoint union of two subsets, and w_I is the ordered exterior product of the w_j for $j \in I$.

We therefore require $u\widehat{\otimes}u$ to be a linear combination of such elements, that is, for all p,

$$\sum_j u_j\widehat{\otimes}u_{p-j} = \sum w_I\widehat{\otimes}w_{I'}.$$

This gives rise to an obvious quadratic constraint: that $u_j \wedge u_k$ should depend only on $j+k$. More precisely taking into account the properties of the graded tensor product we see that for all odd j

$$u_0 u_j = u_1 \wedge u_{j-1},$$

and for all even j

$$u_0 \wedge u_j = u_2 \wedge u_{j-2}.$$

These are just the quadratic equations for pure spinors [12, Sections 92 and 108]. □

Although we have dealt only with the even-dimensional case, the analogous result is easily proved for odd-dimensional V too. It would also be possible to consider pure spinors from the point of view of the action of the spin groups. In that case the even-dimensional spaces decompose into two irreducible pieces on the odd or even part of the exterior algebra. The algebraic constraints on the pure spinors are automatically satisfied in dimensions up to 7. It is the pure spinors which form the most natural generalisation of Penrose's twistors in higher dimensions [44]. It is easy to check that the pure spinors for $SO(2p, 2q)$ form the homogeneous space $\mathrm{Spin}(2p, 2q)/U(p, q)$. (The spinor $1 \in \wedge^0 W \subset \wedge W$ is stabilised by the unitary subgroup, $U(p, q)$ which respects the choice of complex structure.)

11. Quantum groups

Another interesting variation of the basic idea occurs for quantum groups. Consider, for example $U_q(\mathfrak{g})$, for $\mathfrak{g}$ a semi-simple Lie algebra with Cartan matrix a_{jk}, and q a non-zero complex number. (We shall write $q_j = q^{d_j}$ where the coprime numbers d_j are such that $d_j a_{jk}$ is symmetric.) This is the unital Hopf algebra generated by elements K_j, K_j^{-1}, X_j^+ and X_j^- for $j = 1, \ldots, n$, subject to the relations that all the K_j and K_k^{-1} commute with each other, that K_j^{-1} is a two-sided inverse for K_j, and that

$$K_j X_k^{\pm} K_j^{-1} = q^{\pm a_{jk}} X_k^{\pm}, \qquad X_j^+ X_k^- - X_k^- X_j^+ = \delta_{jk}\frac{K_j - K_j^{-1}}{q_j - q_j^{-1}},$$

together with the Serre relations

$$\mathrm{ad}(X_j)^{1-a_{jk}} X_k = 0,$$

where ad denotes the quantum adjoint action.

The comultiplication is given by $\Delta(K_j) = K_j \otimes K_j$, and

$$\Delta(X_j^+) = X_j^+ \otimes K_j + 1 \otimes X_j^+, \qquad \Delta(X_j^-) = X_j^- \otimes 1 + K_j^{-1} \otimes X_j^-.$$

For each vector v in a $U_q(\mathfrak{g})$-module V, one may study the action of $\Delta(U_q(\mathfrak{g}))$ on $v \otimes v$ and minimise the composition series as before. However, this turns out to be rather restrictive and picks out just those vectors, v, which are highest or lowest weights for the preferred Cartan subalgebra generated by the $K_j^{\pm}$. Where possible, it is more sensible to take the q-symmetric tensor product $V \otimes_S V$, which is a quotient of $V \otimes V$ by an invariant subspace, and to look only at the projection of the tensor product into that. It is sufficient that the product, $\hat{R}$, of a flip of the two factors with the universal R matrix (which acts on $V \otimes V$), satisfies a polynomial identity. (This is true for the quantum groups derived from the classical Lie groups.) The image of the product of those polynomial factors which tend to $\hat{R} - 1$ as $q \to 1$, is an analogue of the exterior product, and the quotient by this gives an analogue of the symmetric product. One may then argue as in [22] that repeated application of the $\Delta(X_j^+)$ will eventually give a multiple of the q-symmetric product of a highest weight vector, w, with itself, and $v \otimes_S v$ can be recovered by applying the corresponding product of $\Delta(X_j^-)$ to this, so that v is in the quantum orbit of w.

For example, we take $\mathfrak{g} = \mathfrak{su}(n+1)$, and V to be its natural $(n+1)$-dimensional module, with $v_1, \ldots, v_{n+1}$ a basis of eigenvectors for the preferred Cartan subalgebra, so that

$$K_i v_j = q^{\delta_{ij} - \delta_{ij-1}} v_j, \qquad X_i^+ v_j = \delta_{ij-1} v_{j-1}, \qquad X_i^- v_j = \delta_{ij} v_{j+1}.$$

The subspace of the tensor product generated by the elements $v_j \otimes v_k - q^{-1} v_k \otimes v_j$ with $j < k$ is invariant under the tensor product action of the Hopf algebra, and the quotient gives an analogue of the symmetric product. (One can also realise the symmetric product as a subspace of $V \otimes V$.) Any vector in V turns out to be a dominant vector with this modified definition, whereas only the vectors v_1 and v_{n+1} would qualify if one had not taken the quotient.

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Appendix A. Biholomorphisms of flag manifolds

The main purpose of this appendix is to characterise the biholomorphisms of a flag manifold $G_{\mathbb{C}}/P$, for any parabolic P, a result which is probably well known to geometers but which I have not managed to trace. (Ref. [17] looks at a similar question for non-compact symmetric spaces.) As a first step we look at the Lie algebra of holmorphic vector fields. For standard material on Lie algebras we refer to [6,27].

Theorem A.1. *The holomorphic vector fields on $G_{\mathbb{C}}/P$ are given by $\mathfrak{g}_{\mathbb{C}}$ except when the Dynkin diagram for P is crossed on the first node in the case of B_r $(r > 1)$, on the last node in the case of C_r $(r > 1)$ or G_2, or on the third node in the case of F_4. In these exceptional cases the Lie algebra of holomorphic vector fields is D_{r+1} rather than B_r, A_{2r-1} rather than C_r, C_3 rather than G_2 and E_6 rather than F_4.*

Proof. The global vector fields are given by $H^0(G_{\mathbb{C}}/P, \mathcal{O}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}))$ where $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ is the fibre of the tangent bundle $T(G_{\mathbb{C}}/P)$. Although in some cases it is irreducible [2, Table 3.2], the P-module $\mathfrak{g}/\mathfrak{p}$ is generally given by a composition series. By the Borel–Weil–Bott theorem the global vector fields will have summands $H^0(G_{\mathbb{C}}/P, \cdot)$ with coefficents in the sheaves defined by the composition factors. Each factor can be represented by the Dynkin diagram for the parabolic with the nodes labelled with appropriate weights (which will come from roots in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$). The highest root will automatically give $H^0 \cong \mathfrak{g}_{\mathbb{C}}$, corresponding to the adjoint representation. Since only weights in the fundamental Weyl chamber give non-vanishing H^0, the only chance of obtaining any other non-vanishing terms is for another root (as well as the highest) to lie in the fundamental Weyl chamber. The intersection of the root diagram with the plane spanned by the two roots will define a rank 2 root diagram. Comparison with the known diagrams (e.g. [6, Planche X]) shows that only the diagrams of $B_2 = C_2$ and G_2 have two roots on the edges of the fundamental Weyl chamber. (D_2 decomposes as $A_1 \times A_1$ and so creates no problem, and higher-dimensional groups of this type are also innocuous since they contain $D_3 \cong A_3$ which does not contain such a configuration). The Lie algebras of G_2 and F_4 and of B_r, C_r for $r \geq 2$ have Dynkin diagrams containing those of $B_2 = C_2$ or G_2, and have more than one root in the fundamental Weyl chamber. The problem certainly arises for a minimal parabolic of any of these algebras since each root spans a one-dimensional composition factor, and all those in the fundamental Wey! chamber contribute summands to the Lie algebra of holomoprhic vector fields.

More detailed analysis enables us to specify which parabolics give problems. We shall choose P to be the stabiliser of a lowest weight vector in the normal sense, so that its Lie algebra $\mathfrak{p}$ contains all the negative root vectors. It also contains those positive root vectors which are in the kernel of the lowest weight (or Killing orthogonal to it, when $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified), which together with their positive counterparts and some elements of the Cartan subalgebra, give the reductive component of the Levi decomposition. The remaining (non-orthogonal) positive roots, including those in the fundamental Weyl chamber, define a nilpotent algebra, which can be identified with $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$.

Using the notation of Bourbaki [6, Planche II] for B_r, both ε_1 and the highest root $\varepsilon_1+\varepsilon_2$ are in the fundamental Weyl chamber. For the fundamental weights $\varpi_k=\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k$, with $k<r$ one sees that $\varepsilon_1\pm\varepsilon_r$ are also in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$, whilst $\pm\varepsilon_r$ are orthogonal to ϖ and so in $\mathfrak{p}$ ensure that they are in the same composition factor. Since ε_1 is less positive than $\varepsilon_1+\varepsilon_r$ it cannot be the highest weight of that factor. This removes the potential problem. For $\varpi_r=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_r)$, however, the reductive part of $\mathfrak{p}$ is the A_{r-1} algebra generated by the roots $\varepsilon_i-\varepsilon_j$ for $i\neq j$. The roots in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ are ε_j which span a composition factor with highest weight ε_1 and $\varepsilon_i+\varepsilon_j$ for $i\neq j$ which span a composition factor with highest weight $\varepsilon_1+\varepsilon_2$. So in this case, whose Dynkin diagram has just the final node crossed, the problem does arise. Since the parabolics for combinations of the fundamental weights are the intersection of those for the individual weights the problem will also arise for any parabolic which has the final node crossed. The highest root always gives the adjoint representation, whilst ε_1 is highest weight for the natural $(2r+1)$-dimensional representation of $B_r=SO(2r+1)$. The holomorphic vector fields are given by the sum of these, which is just the Lie algebra of $D_{r+1}=SO(2r+2)$.

For C_r [6, Planche III], where both $2\varepsilon_1$ and $\varepsilon_1+\varepsilon_2$ are in the fundamental chamber, there is no problem for the fundamental weights $\varpi_k=\varepsilon_1+\cdots+\varepsilon_k$ when $k>1$, because $\mathfrak{p}$ contains $\pm(\varepsilon_1-\varepsilon_2)$, so that $\varepsilon_1+\varepsilon_2$ is always in the same composition factor as the highest root $2\varepsilon_1$. However, for $\varpi_1=\varepsilon_1$ the two occur as highest weights of different composition factors. This corresponds to a Dynkin diagram with the first node crossed. Whilst the highest root gives the adjoint representation $\varepsilon_1+\varepsilon_2$ gives the second exterior power of the natural $2r$-dimensional representation arising from the identification of $C_r=Sp(2r)$ with the subgroup $Sp(2r,\mathbb{C})\cap SU(2r)$ of $SU(2r)$. The direct sum of these two Lie algebra of $A_{2r-1}=SU(2r)$.

The exceptional algebra G_2 [6, Planche IX] has only two fundamental weights which are both roots in the fundamental Weyl chamber. The adjoint representation has highest weight ϖ_2. The composition factors for $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ have highest weights ϖ_2 and $3\varpi_1-\varpi_2$, so only one of these contributes to the holomorphic vector fields. The other fundamental weight ϖ_1, which gives the natural seven-dimensional representation, gives composition factors with highest weights ϖ_1, ϖ_2 and $\varpi_2-\varpi_1$, of which the first two survive leading to a problem in this case. The Lie algebra of holomorphic vector fields is the direct sum of the adjoint representation coming from ϖ_2 and the seven-dimensional representation coming from ϖ_1, which together give the Lie algebra of $C_3=Sp(6)$.

Finally, F_4 [6, Planche VIII] resembles B_r. The two roots in the fundamental Weyl chamber are $\varepsilon_1+\varepsilon_2$ and ε_1. For the fundamental weights ϖ_j with $j\neq 3$ the reductive part of $\mathfrak{p}$ contains $\pm\varepsilon_4$ which means that ε_1 is in the same composition factor as the higher root $\varepsilon_1+\varepsilon_4$, and there is no problem. For ϖ_3, however, ε_1 and $\varepsilon_1+\varepsilon_2$ both occur as highest weights of composition factors. (The dimension of the associated irreducible representation is 273, and the stabiliser of the highest weight is $S(U(3)\times U(2))$, so that the orbit is $F_4/S(U(3)\times U(2))$.) Whilst $\varpi_1=\varepsilon_1+\varepsilon_2$ gives the adjoint representation, $\varpi_4=\varepsilon_1$ gives the 26-dimensional irreducible, and their direct sum gives the Lie algebra of E_6. This completes the proof. □

The corresponding highest weight orbits can be described quite explicitly, for example, the following result describes the situation for maximal parabolic subgroups, and clearly we can follow the same principles to construct examples for smaller parabolics.

Theorem A.2. *In the first three exceptional cases when the highest weight vector is stabilised by a maximal parabolic subgroup its projective orbit is given by*

$$\begin{aligned} \mathrm{Spin}(2r+1)/U(r) &= \mathrm{Spin}(2r+2)/U(r+1), \\ Sp(2r)/(U(1)\times Sp(2r-2)) &= SU(2r)/U(2r-1), \\ G_2/U(2) &= SO(7)/(SO(2)\times SO(5)). \end{aligned}$$

Proof. We take the representation whose fundamental weight corresponds to the single crossed node of the maximal parabolic. In the case of $B_r \sim \mathrm{Spin}(2r+1)$ this is the spin representation (since we are interested only in the projective action it does not matter that this is a projective representation). This can be extended to one of the two spin representations of $D_{r+1} \sim \mathrm{Spin}(2r+2)$. The ray through a highest weight vector for $\mathrm{Spin}(2r+2)$ is stabilised by $U(r+1)$. $\mathrm{Spin}(2r+1)$ acts transitively on its orbit, with stabiliser $\mathrm{Spin}(2r+1)\cap U(r+1) = U(r)$, so that the orbit is given by $\mathrm{Spin}(2r+2)/U(r+1) = \mathrm{Spin}(2r+1)/U(r)$. For $Sp(2r) = Sp(2r, \mathbb{C})\cap SU(2r)$ we take the natural $2r$-dimensional representation. A highest weight vector for $SU(2r)$ generates a ray stabilised by $U(2r-1)$. $Sp(2r)$ acts transitively on its orbit with stabiliser $Sp(2r)\cap U(2r-1) = U(1)\times Sp(2r-2)$, giving the second identity. In the final case we take a highest weight ray in the spin representation of $SO(7)$, which has stabiliser $SO(2)\times SO(5)$. As noted in [18, p. 391] the highest weight vectors for G_2 also lie on a $SO(7)$-invariant quadric, which must be the same as the quadric of $SO(7)$ highest weight vectors, and can be expressed as $G_2/U(2)$ for a $U(2)$ generated by the torus of G_2 and two long roots (see also [24, Chapter 6, Problem 9(b)]). The third identity now follows. □

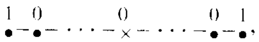
The dimension of the irreducible representation of F_4 with highest weight ϖ_3 is 273, so that it does not extend to a fundamental representation of E_6. Moreover, the stabiliser of the highest weight is $S(U(3)\times U(2))$, so that the orbit is $F_4/S(U(3)\times U(2))$. None of the parabolic subgroups of E_6 gives a quotient of this dimension so that we are not dealing with a flag manifold.

We are now in a position to characterise the full group of biholomorphisms of the flag manifolds. Let us first note that certain automorphisms of the Lie algebras and Lie groups arise from symmetries of the Dynkin diagram. Thus A_r has a reflection symmetry in which the order of nodes is reversed, E_6 has a similar reflection symmetry of the horizontal nodes, for $r \neq 3$, D_r has an involutive symmetry interchanging the two tails, whilst D_3 has S_3 symmetry permuting its three tails. When a parabolic subgroup P is chosen some nodes are crossed and generally only a subgroup R_P of this symmetry group will respect this pattern.

Theorem A.3. *Except in the cases described in Theorem A.1 the group of biholomorphisms of a flag manifold $G_{\mathbb{C}}/P$ is the semi-direct product $R_P\cdot G_{\mathbb{C}}$. In the exceptional cases one takes the larger group given in Theorem A.2.*

Proof. Any biholomorphism of $G_{\mathbb{C}}/P$ will define an automorphism of the Lie algebra of holomorphic vector fields, which in the non-exceptional cases, is $\mathfrak{g}_{\mathbb{C}}$. The automorphism group of a Lie algebra is given by automorphisms of its Dynkin diagram (see e.g. [18, Proposition D.40]). Moreover, the automorphisms of $G_{\mathbb{C}}/P$ must also respect the parabolic subgroup up to conjugacy. This gives R_P. In the exceptional cases we simply use the realisation of the flag manifold using the larger group and parabolic, as this never gives an exception. □

Example. Consider the case of $G = A_{n-1} = SU(n)$ and the maximal parabolic given in the notation of [2] by



(the notation of Fulton and Harris [18] is similar but uses different symbols instead of • and ×). If the cross is on the kth node then the corresponding orbit of the highest weight vector is the Grassmannian $Gr_k(n) = SU(n)/S(U(k) \times U(n-k))$. According to our results the group of biholomorphisms is $G_{\mathbb{C}} = SL(n, \mathbb{C})$, unless $n = 2k$ (so that the middle node of $2n - 1$ is crossed), in which case it acquires an extra symmetry of order 2 (implementing the reflection symmetry of the Dynkin diagram when crossed in the middle, which geometrically is the Hodge duality). Thus we recover Chow's theorem on the biholomorphisms of Grassmannians [13]. His results on other algebraic symmetric spaces can similarly be deduced. (Chow's work considers for the classical groups special cases in which only one node is crossed.)

A concrete example of this can be seen for the representation of $SU(4)$ on $V = \wedge^2\mathbb{C}^4$, where the dominant vectors are defined by the single Klein quadratic of signature (3,3). The subgroup of $GL(V)$ preserving dominant vectors is the complex orthogonal group $O(3, 3; \mathbb{C})$ whose connected component $SO(3, 3; \mathbb{C})$ is the complexification of $SU(4)/\{\pm 1\}$ (the central scalars make no difference to the projective action).

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